

Superstring and Superstring Field Theory: A New Solution Using Ultradistributions of Exponential Type

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Abstract In this paper we show that Ultradistributions of Exponential Type (UET) are appropriate for the description in a consistent way superstring and superstring field theories. A new Lagrangian for the closed superstring is given. We show that the superstring field is a linear superposition of UET of compact support, and give the notion of anti-superstring. We evaluate the propagator for the superstring field, and calculate the convolution of two of them.

Keywords Ultradistributions · Field theory · String field theory

1 Introduction

In a series of papers [1–5] we have shown that Ultradistribution theory of Sebastiao e Silva [6–8] permits a significant advance in the treatment of quantum field theory. In particular, with the use of the convolution of Ultradistributions we have shown that it is possible to define a general product of distributions (a product in a ring with divisors of zero) that sheds new light on the question of the divergences in Quantum Field Theory. Furthermore, Ultradistributions of Exponential Type (UET) are adequate for to describe Gamow States and exponentially increasing fields in Quantum Field Theory [9–11].

Ultradistributions also have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with and, as we shall see, have interesting properties. One of those properties is that Schwartz's tempered distributions are canonical and continuously injected into UET and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with Ultradistributions of Exponential Type.

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Another interesting property is that the space of UET is reflexive under the operation of Fourier transform (in a way similar to that of tempered distributions of Schwartz).

In a recent paper [12] we have shown that Ultradistributions of Exponential type provides an adequate framework for a consistent treatment of string and string field theories. In particular, a general state of the closed bosonic bradyonic string is represented by UET of compact support, and as a consequence the string field of a bradyonic bosonic string is a linear combination of UET of compact support (CUET).

In this paper we extend the formalism developed in [12] to the supersymmetric string.

This paper is organized as follows: in Sects. 2 and 3 we define the Ultradistributions of Exponential Type and their Fourier transform. They are part of a Guelfand’s Triplet (or Rigged Hilbert Space [13]) together with their respective duals and a “middle term” Hilbert space. In Sects. 4 and 5 we give the main results obtained in [12] to be applied in this paper. In Sect. 6 we treat the supersymmetric string, giving a new Lagrangian, defining the physical state of the string and solving the non-linear Euler-Lagrange equations and the constraints. In Sect. 7 we give a representation for the states of the string using CUET of compact support. Also we obtain the expression for a general state of the supersymmetric string. In Sect. 8 we give expressions for the field of the string, the string field propagator and the creation and annihilation operators of a string. We define in an analog way to Quantum Field Theory the notion of anti-string and its corresponding creation and annihilation fields. In Sect. 9, we give expressions for the non-local action of a free superstring and a non-local interaction Lagrangian for the string field inspired in Quantum Field Theory. Also we show how to evaluate the convolution of two superstring field propagators. Finally, Sect. 10 is reserved for a discussion of the principal results.

2 Ultradistributions of Exponential Type

Let \mathcal{S} be the Schwartz space of rapidly decreasing test functions. Let Λ_j be the region of the complex plane defined as:

$$\Lambda_j = \{z \in \mathbb{C} : |\Im(z)| < j : j \in \mathbb{N}\} \tag{2.1}$$

According to [6, 8] the space of test functions $\hat{\phi} \in \mathcal{V}_j$ is constituted by all entire analytic functions of \mathcal{S} for which

$$\|\hat{\phi}\|_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{(j|\Re(z)|)} |\hat{\phi}^{(k)}(z)| \right] \right\} \tag{2.2}$$

is finite.

The space \mathcal{Z} is then defined as:

$$\mathcal{Z} = \bigcap_{j=0}^{\infty} \mathcal{V}_j \tag{2.3}$$

It is a complete countably normed space with the topology generated by the system of seminorms $\{\|\cdot\|_j\}_{j \in \mathbb{N}}$. The dual of \mathcal{Z} , denoted by \mathfrak{B} , is by definition the space of ultradistributions of exponential type [6, 8]. Let \mathfrak{S} be the space of rapidly decreasing sequences. According to [13] \mathfrak{S} is a nuclear space. We consider now the space of sequences \mathfrak{P} generated by the

Taylor development of $\hat{\phi} \in \mathfrak{Z}$

$$\mathfrak{P} = \left\{ \mathfrak{Q} : \mathfrak{Q} \left(\hat{\phi}(0), \hat{\phi}'(0), \frac{\hat{\phi}''(0)}{2}, \dots, \frac{\hat{\phi}^{(n)}(0)}{n!}, \dots \right) : \hat{\phi} \in \mathfrak{Z} \right\} \tag{2.4}$$

The norms that define the topology of \mathfrak{P} are given by:

$$\|\hat{\phi}\|'_p = \sup_n \frac{n^p}{n} |\hat{\phi}^n(0)| \tag{2.5}$$

\mathfrak{P} is a subspace of \mathfrak{S} and therefore is a nuclear space. As the norms $\|\cdot\|_j$ and $\|\cdot\|'_p$ are equivalent, the correspondence

$$\mathfrak{Z} \iff \mathfrak{P} \tag{2.6}$$

is an isomorphism and therefore \mathfrak{Z} is a countably normed nuclear space. We can define now the set of scalar products

$$\begin{aligned} \langle \hat{\phi}(z), \hat{\psi}(z) \rangle_n &= \sum_{q=0}^n \int_{-\infty}^{\infty} e^{2n|z|} \overline{\hat{\phi}^{(q)}(z)} \hat{\psi}^{(q)}(z) dz \\ &= \sum_{q=0}^n \int_{-\infty}^{\infty} e^{2n|x|} \overline{\hat{\phi}^{(q)}(x)} \hat{\psi}^{(q)}(x) dx \end{aligned} \tag{2.7}$$

This scalar product induces the norm

$$\|\hat{\phi}\|''_n = [\langle \hat{\phi}(x), \hat{\phi}(x) \rangle_n]^{\frac{1}{2}} \tag{2.8}$$

The norms $\|\cdot\|_j$ and $\|\cdot\|''_n$ are equivalent, and therefore \mathfrak{Z} is a countably Hilbertian nuclear space. Thus, if we call now \mathfrak{Z}_p the completion of \mathfrak{Z} by the norm p given in (2.8), we have:

$$\mathfrak{Z} = \bigcap_{p=0}^{\infty} \mathfrak{Z}_p \tag{2.9}$$

where

$$\mathfrak{Z}_0 = H \tag{2.10}$$

is the Hilbert space of square integrable functions.

As a consequence the “nested space”

$$\mathfrak{A} = (\mathfrak{Z}, H, \mathfrak{B}) \tag{2.11}$$

is a Guelfand’s triplet (or a Rigged Hilbert space = RHS. See [13]).

Any Guelfand’s triplet $\mathfrak{G} = (\Phi, H, \Phi')$ has the fundamental property that a linear and symmetric operator on Φ , admitting an extension to a self-adjoint operator in H , has a complete set of generalized eigen-functions in Φ' with real eigenvalues.

\mathfrak{B} can also be characterized in the following way [6, 8]: let \mathfrak{C}_ω be the space of all functions $\hat{F}(z)$ such that:

- (I) $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C} : |\text{Im}(z)| > p\}$.

(II) $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is bounded continuous in $\{z \in \mathbb{C} : |\text{Im}(z)| \geq p\}$, where $p = 0, 1, 2, \dots$ depends on $\hat{F}(z)$.

Let \mathfrak{H} be: $\mathfrak{H} = \{\hat{F}(z) \in \mathfrak{E}_\omega : \hat{F}(z) \text{ is entire analytic}\}$. Then \mathfrak{B} is the quotient space:
 (III) $\mathfrak{B} = \mathfrak{E}_\omega / \mathfrak{H}$.

Due to these properties it is possible to represent any ultradistribution as [6, 8]:

$$\hat{F}(\hat{\phi}) = \langle \hat{F}(z), \hat{\phi}(z) \rangle = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz \tag{2.12}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $\text{Im}(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\text{Im}(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$.)

Formula (2.12) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of ‘‘Dirac formula’’ for exponential ultradistributions [6]:

$$\hat{F}(z) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{t - z} dt \equiv \frac{\cosh(\lambda z)}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t - z) \cosh(\lambda t)} dt \tag{2.13}$$

where the ‘‘density’’ $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz = \int_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t) dt \tag{2.14}$$

equation (2.13) should be used carefully. While $\hat{F}(z)$ is analytic on Γ , the density $\hat{f}(t)$ is in general singular, so that the r.h.s. of (2.14) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ , $\hat{F}(z)$ is bounded by an exponential and a power of z [6, 8]:

$$|\hat{F}(z)| \leq C|z|^p e^{p|\Re(z)|} \tag{2.15}$$

where C and p depend on \hat{F} .

The representation (2.12) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{H}$ to $\hat{F}(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z) dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz + \oint_{\Gamma} \hat{G}(z)\hat{\phi}(z) dz$$

But:

$$\oint_{\Gamma} \hat{G}(z)\hat{\phi}(z) dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is entire analytic (and rapidly decreasing),

$$\therefore \oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z) dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz \tag{2.16}$$

Another very important property of \mathfrak{B} is that \mathfrak{B} is reflexive under the Fourier transform:

$$\mathfrak{B} = \mathcal{F}_c \{\mathfrak{B}\} = \mathcal{F} \{\mathfrak{B}\} \tag{2.17}$$

where the complex Fourier transform $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by:

$$\begin{aligned} F(k) &= \Theta[\Im(k)] \int_{\Gamma_+} \hat{F}(z)e^{ikz} dz - \Theta[-\Im(k)] \int_{\Gamma_-} \hat{F}(z)e^{ikz} dz \\ &= \Theta[\Im(k)] \int_0^\infty \hat{f}(x)e^{ikx} dx - \Theta[-\Im(k)] \int_{-\infty}^0 \hat{f}(x)e^{ikx} dx \end{aligned} \tag{2.18}$$

Here Γ_+ is the part of Γ with $\Re(z) > 0$ and Γ_- is the part of Γ with $\Re(z) < 0$. Using (2.18) we can interpret Dirac’s formula as:

$$F(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{f(s)}{s - k} ds \equiv \mathcal{F}_c \{ \mathcal{F}^{-1} \{ f(s) \} \} \tag{2.19}$$

The treatment for ultradistributions of exponential type defined on \mathbb{C}^n is similar to the case of one variable. Thus

$$\Lambda_j = \{ z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |\Im(z_k)| \leq j, 1 \leq k \leq n \} \tag{2.20}$$

$$\|\hat{\phi}\|_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{j|\sum_{p=1}^n |\Re(z_p)|} \left| D^{(k)} \hat{\phi}(z) \right| \right] \right\} \tag{2.21}$$

where $D^{(k)} = \partial^{(k_1)} \partial^{(k_2)} \dots \partial^{(k_n)}$, $k = k_1 + k_2 + \dots + k_n$.

\mathfrak{B}^n is characterized as follows. Let \mathfrak{E}_ω^n be the space of all functions $\hat{F}(z)$ such that:

- (I)' $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C}^n : |\Im(z_1)| > p, |\Im(z_2)| > p, \dots, |\Im(z_n)| > p\}$.
- (II)' $\hat{F}(z)e^{-[p \sum_{j=1}^n |\Re(z_j)|]} / z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |\Im(z_1)| \geq p, |\Im(z_2)| \geq p, \dots, |\Im(z_n)| \geq p\}$, where $p = 0, 1, 2, \dots$ depends on $\hat{F}(z)$.

Let \mathfrak{F}^n be: $\mathfrak{F}^n = \{ \hat{F}(z) \in \mathfrak{E}_\omega^n : \hat{F}(z)$ is entire analytic at minus in one of the variables $z_j \ 1 \leq j \leq n \}$ Then \mathfrak{B}^n is the quotient space:

- (III)' $\mathfrak{B}^n = \mathfrak{E}_\omega^n / \mathfrak{F}^n$. We have now

$$\hat{F}(\hat{\phi}) = (\hat{F}(z), \hat{\phi}(z)) = \oint_{\Gamma} \hat{F}(z) \hat{\phi}(z) dz_1 dz_2 \dots dz_n \tag{2.22}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$ where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\Im(z_j) > \zeta, \zeta > p$ and back from ∞ to $-\infty$ for $\Im(z_j) < -\zeta, -\zeta < -p$. (Again Γ surrounds all the singularities of $\hat{F}(z)$.) The n -dimensional Dirac’s formula is

$$\hat{F}(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^\infty \frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2) \dots (t_n - z_n)} dt_1 dt_2 \dots dt_n \tag{2.23}$$

where the “density” $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z) \hat{\phi}(z) dz_1 dz_2 \dots dz_n = \int_{-\infty}^\infty f(t) \hat{\phi}(t) dt_1 dt_2 \dots dt_n \tag{2.24}$$

and the modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C |z|^p e^{[p \sum_{j=1}^n |\Re(z_j)|]} \tag{2.25}$$

where C and p depend on \hat{F} .

3 The Case $N \rightarrow \infty$

When the number of variables of the argument of the Ultradistribution of Exponential type tends to infinity we define:

$$d\mu(x) = \frac{e^{-x^2}}{\sqrt{\pi}} dx \tag{3.1}$$

If $\hat{\phi}(x_1, x_2, \dots, x_n)$ is such that:

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |\hat{\phi}(x_1, x_2, \dots, x_n)|^2 d\mu_1 d\mu_2 \dots d\mu_n < \infty \tag{3.2}$$

where

$$d\mu_i = \frac{e^{-x_i^2}}{\sqrt{\pi}} dx_i \tag{3.3}$$

Then by definition $\hat{\phi}(x_1, x_2, \dots, x_n) \in L_2(\mathbb{R}^n, \mu)$ and

$$L_2(\mathbb{R}^\infty, \mu) = \bigcup_{n=1}^{\infty} L_2(\mathbb{R}^n, \mu) \tag{3.4}$$

Let $\hat{\psi}$ be given by

$$\hat{\psi}(z_1, z_2, \dots, z_n) = \pi^{n/4} \hat{\phi}(z_1, z_2, \dots, z_n) e^{\frac{z_1^2 + z_2^2 + \dots + z_n^2}{2}} \tag{3.5}$$

where $\hat{\phi} \in \mathfrak{Z}^n$ (the corresponding n -dimensional of \mathfrak{Z}).

Then by definition $\hat{\psi}(z_1, z_2, \dots, z_n) \in \mathfrak{G}(\mathbb{C}^n)$,

$$\mathfrak{G}(\mathbb{C}^\infty) = \bigcup_{n=1}^{\infty} \mathfrak{G}(\mathbb{C}^n) \tag{3.6}$$

and the dual $\mathfrak{G}'(\mathbb{C}^\infty)$ given by

$$\mathfrak{G}'(\mathbb{C}^\infty) = \bigcup_{n=1}^{\infty} \mathfrak{G}'(\mathbb{C}^n) \tag{3.7}$$

is the space of Ultradistributions of Exponential type.

The analog to (2.11) in the infinite dimensional case is:

$$\mathfrak{W} = (\mathfrak{G}(\mathbb{C}^\infty), L_2(\mathbb{R}^\infty, \mu), \mathfrak{G}'(\mathbb{C}^\infty)) \tag{3.8}$$

If we define:

$$\mathcal{F} : \mathfrak{G}(\mathbb{C}^\infty) \rightarrow \mathfrak{G}(\mathbb{C}^\infty) \tag{3.9}$$

via the Fourier transform:

$$\mathcal{F} : \mathfrak{G}(\mathbb{C}^n) \rightarrow \mathfrak{G}(\mathbb{C}^n) \tag{3.10}$$

given by:

$$\mathcal{F}\{\hat{\psi}\}(k) = \int_{-\infty}^{\infty} \hat{\psi}(z_1, z_2, \dots, z_n) e^{ik \cdot z + \frac{k^2}{2}} d\rho_1 d\rho_2 \dots d\rho_n \tag{3.11}$$

where

$$d\rho(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \tag{3.12}$$

we conclude that

$$\mathfrak{G}'(\mathbb{C}^\infty) = \mathcal{F}_c\{\mathfrak{G}'(\mathbb{C}^\infty)\} = \mathcal{F}\{\mathfrak{G}'(\mathbb{C}^\infty)\} \tag{3.13}$$

where in the one-dimensional case

$$\mathcal{F}_c\{\hat{\psi}\}(k) = \Theta[\Im(k)] \int_{\Gamma_+} \hat{\psi}(z) e^{ikz + \frac{k^2}{2}} d\rho - \Theta[-\Im(k)] \int_{\Gamma_-} \hat{\psi}(z) e^{ikz + \frac{k^2}{2}} d\rho \tag{3.14}$$

4 The Constraints for a Bradyonic Bosonic String

The constraints for a bradyonic bosonic string have been deduced in [12]. We can describe the bosonic string by a system composed of a Lagrangian, one constraint and two initial conditions:

$$\begin{cases} \mathcal{L} = |\dot{X}^2 - X'^2| \\ (\dot{X} + X')^2 = 0 \\ X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0 \end{cases} \tag{4.1}$$

or equivalently

$$\begin{cases} \mathcal{L} = |\dot{X}^2 - X'^2| \\ (\dot{X} - X')^2 = 0 \\ X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0 \end{cases} \tag{4.2}$$

5 A Representation for the States of the Closed Bosonic String

The Case n Finite

From [12] we have

$$a = -z; \quad a^+ = \frac{d}{dz} \tag{5.1}$$

and then

$$[a, a^+] = 1 \tag{5.2}$$

Thus we have a representation for creation and annihilation operators of the states of the string. The vacuum state annihilated by z_μ is the UET $\delta(z_\mu)$, and the orthonormalized states

obtained by successive application of $\frac{d}{dz_\mu}$ to $\delta(z_\mu)$ are:

$$F_n(z_\mu) = \frac{\delta^{(n)}(z_\mu)}{\sqrt{n!}} \tag{5.3}$$

A general state of the string can be written as:

$$\begin{aligned} \phi(x, \{z\}) = & [a_0(x) + a_{\mu_1}^{i_1}(x)\partial_{i_1}^{\mu_1} + a_{\mu_1\mu_2}^{i_1i_2}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2} + \dots + \dots \\ & + a_{\mu_1\mu_2\dots\mu_n}^{i_1i_2\dots i_n}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2}\dots\partial_{i_n}^{\mu_n} + \dots + \dots] \delta(\{z\}) \end{aligned} \tag{5.4}$$

where $\{z\}$ denotes $(z_{1\mu}, z_{2\mu}, \dots, z_{n\mu}, \dots, \dots)$, and ϕ is a UET of compact support in the set of variables $\{z\}$. The functions $a_{\mu_1\mu_2\dots\mu_n}^{i_1i_2\dots i_n}(x)$ are solutions of

$$\square a_{\mu_1\mu_2\dots\mu_n}^{i_1i_2\dots i_n}(x) = 0 \tag{5.5}$$

The Case $n \rightarrow \infty$

In this case

$$a = -z; \quad a^+ = -2z + \frac{d}{dz} \tag{5.6}$$

we have

$$[a, a^+] = 1 \tag{5.7}$$

The vacuum state annihilated by a is $\delta(z)e^{z^2}$. The orthonormalized states obtained by successive application of a^+ are:

$$\hat{F}_n(z) = 2^{\frac{1}{4}}\pi^{\frac{1}{2}} \frac{\delta^{(n)}(z)e^{z^2}}{\sqrt{n!}} \tag{5.8}$$

6 The Supersymmetric String

According to the treatment given in [12] the action for the supersymmetric string is given by:

$$S = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\pi |\dot{\Pi}^2 - \Pi'^2| d^2\sigma \tag{6.1}$$

where

$$\begin{aligned} \dot{\Pi}^\mu &= \dot{X}^\mu + \frac{i}{2}\bar{\Theta}\Gamma^\mu\Theta - \frac{i}{2}\bar{\Theta}'\Gamma^\mu\dot{\Theta} \\ \Pi'^\mu &= X'^\mu + \frac{i}{2}\bar{\Theta}\Gamma^\mu\Theta - \frac{i}{2}\bar{\Theta}'\Gamma^\mu\Theta' \end{aligned} \tag{6.2}$$

(see [14]) and Θ is a Dirac spinor.

We define

$$\dot{X}_\infty^\mu = \lim_{\tau \rightarrow \infty} \dot{X}^\mu(\tau, \sigma) \tag{6.3}$$

Following [12] two possible set of constraints for the string are:

$$\begin{cases} (\dot{\Pi} + \Pi')^2 = 0 \\ \Gamma \cdot \dot{X}_\infty = 0 \end{cases} \tag{6.4}$$

$$\begin{cases} (\dot{\Pi} - \Pi')^2 = 0 \\ \Gamma \cdot \dot{X}_\infty = 0 \end{cases} \tag{6.5}$$

Thus, we have that to solve the system described by:

$$\begin{cases} \mathcal{L} = |\dot{\Pi}^2 - \Pi'^2| \\ (\dot{\Pi} + \Pi')^2 = 0 \\ \Gamma \cdot \dot{X}_\infty = 0 \\ X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0 \\ \Theta_\mu(\tau, 0) = \Theta_\mu(\tau, \pi) = 0 \end{cases} \tag{6.6}$$

or equivalently

$$\begin{cases} \mathcal{L} = |\dot{\Pi}^2 - \Pi'^2| \\ (\dot{\Pi} - \Pi')^2 = 0 \\ \Gamma \cdot \dot{X}_\infty = 0 \\ X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0 \\ \Theta_\mu(\tau, 0) = \Theta_\mu(\tau, \pi) = 0 \end{cases} \tag{6.7}$$

We define

$$\mathcal{L}_1 = \dot{\Pi}^2 - \Pi'^2 \tag{6.8}$$

Then the Euler-Lagrange equations for (6.6), (6.7) are

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left[\text{Sgn}(\mathcal{L}_1) \left(\dot{X}^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} \right) \right] \\ & - \frac{\partial}{\partial \sigma} \left[\text{Sgn}(\mathcal{L}_1) \left(X'^\mu + \frac{i}{2} \bar{\Theta}' \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta' \right) \right] = 0 \end{aligned} \tag{6.9}$$

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left[\text{Sgn}(\mathcal{L}_1) \left(\dot{X}^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} \right) (\Theta^{+\beta} \Omega_{\beta\alpha}^\mu) \right] \\ & - \frac{\partial}{\partial \sigma} \left[\text{Sgn}(\mathcal{L}_1) \left(X'^\mu + \frac{i}{2} \bar{\Theta}' \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta' \right) (\Theta^{+\beta} \Omega_{\beta\alpha}^\mu) \right] \\ & + \text{Sgn}(\mathcal{L}_1) \left(\dot{X}^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} \right) (\dot{\Theta}^{+\beta} \Omega_{\beta\alpha}^\mu) \\ & - \text{Sgn}(\mathcal{L}_1) \left(X'^\mu + \frac{i}{2} \bar{\Theta}' \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta' \right) (\Theta'^{+\beta} \Omega_{\beta\alpha}^\mu) = 0 \end{aligned} \tag{6.10}$$

$$\begin{aligned}
 & \frac{\partial}{\partial \tau} \left[\text{Sgn}(\mathcal{L}_1) \left(\dot{X}^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} \right) (\Omega_{\alpha\beta}^\mu \Theta^\beta) \right] \\
 & - \frac{\partial}{\partial \sigma} \left[\text{Sgn}(\mathcal{L}_1) \left(X'^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta' \right) (\Omega_{\alpha\beta}^\mu \Theta^\beta) \right] \\
 & + \text{Sgn}(\mathcal{L}_1) \left(\dot{X}^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} \right) (\Omega_{\alpha\beta}^\mu \dot{\Theta}^\beta) \\
 & - \text{Sgn}(\mathcal{L}_1) \left(X'^\mu + \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta' \right) (\Omega_{\alpha\beta}^\mu \Theta'^\beta) = 0
 \end{aligned} \tag{6.11}$$

where $\Omega^\mu = \Gamma^0 \Gamma^\mu$.

The solution for (6.6) are:

$$\begin{cases} \dot{X}^\mu = \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta + \dot{V}^\mu \\ \dot{\Theta} + \Theta' = 0 \\ \dot{V}^\mu + V'^\mu = 0 \end{cases} \tag{6.12}$$

and for (6.7):

$$\begin{cases} \dot{X}^\mu = \frac{i}{2} \bar{\Theta} \Gamma^\mu \dot{\Theta} - \frac{i}{2} \bar{\Theta} \Gamma^\mu \Theta + \dot{V}^\mu \\ \dot{\Theta} - \Theta' = 0 \\ \dot{V}^\mu - V'^\mu = 0 \end{cases} \tag{6.13}$$

From (6.12) we obtain

$$\Theta^\alpha = \sum_{n=0}^{\infty} c_n^\alpha e^{-2in(\tau-\sigma)} + d_n^{+\alpha} e^{2in(\tau-\sigma)} \tag{6.14}$$

$$\begin{aligned}
 V^\mu &= x^\mu - \sum_{n=1}^{\infty} 2n (d_n^\alpha \Omega_{\alpha\beta}^\mu d_n^{+\beta} - c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_n^\beta) \sigma \\
 &+ l^2 p_\mu + \sum_{n=1}^{\infty} 2n (d_n^\alpha \Omega_{\alpha\beta}^\mu d_n^{+\beta} - c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_n^\beta) \tau \\
 &+ \frac{il}{2} \sum_{n=-\infty; n \neq 0}^{\infty} \frac{a_n^\mu}{n} e^{-2in(\tau-\sigma)}
 \end{aligned} \tag{6.15}$$

$$\begin{aligned}
 X^\mu &= x^\mu + l^2 p^\mu \tau + \frac{il}{2} \sum_{s=-\infty; s \neq 0}^{\infty} \frac{a_s^\mu}{s} e^{-2is(\tau-\sigma)} \\
 &+ \sum_{s=-\infty; s \neq 0}^{\infty} \sum_{n=0; n+s \geq 0}^{\infty} i \frac{2n+s}{2s} c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_{n+s}^\beta e^{-2is(\tau-\sigma)} \\
 &+ \sum_{s=-\infty; s \neq 0}^{-1} \sum_{n=0; n+s \leq -1}^{\infty} i \frac{2n+s}{2s} c_n^{+\alpha} \Omega_{\alpha\beta}^\mu d_{-n-s}^{+\beta} e^{-2is(\tau-\sigma)}
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{s=1}^{\infty} \sum_{n=1; s-n \geq 0}^{\infty} i \frac{2n-s}{2s} d_n^\alpha \Omega_{\alpha\beta}^\mu c_{s-n}^\beta e^{-2is(\tau-\sigma)} \\
 & - \sum_{s=-\infty}^{\infty} \sum_{n=1; n-s \geq 1}^{\infty} i \frac{2n-s}{2s} d_n^\alpha \Omega_{\alpha\beta}^\mu d_{n-s}^{+\beta} e^{-2is(\tau-\sigma)}
 \end{aligned} \tag{6.16}$$

Using these solutions, (6.10) and (6.11) transform into:

$$\dot{\Theta}^{+\beta} \Omega_{\beta\alpha}^\mu P_\mu = 0 \tag{6.17}$$

$$p_\mu \Omega_{\alpha\beta}^\mu \Theta^\beta = 0 \tag{6.18}$$

which are consistent with the constraints for (6.6), namely:

$$\begin{cases} p^2 |\Psi\rangle = (\Gamma \cdot p)^2 |\Psi\rangle = 0 \\ \Gamma \cdot p |\Psi\rangle = 0 \end{cases} \tag{6.19}$$

where $|\Psi\rangle$ is the physical state of the string. It is sufficient to solve the second constraint because it implies the first one.

Similarly from (6.13) we obtain

$$\Theta^\alpha = \sum_{n=0}^{\infty} c_n^\alpha e^{-2in(\tau+\sigma)} + d_n^{+\alpha} e^{2in(\tau+\sigma)} \tag{6.20}$$

$$\begin{aligned}
 V^\mu &= x^\mu + \sum_{n=1}^{\infty} 2n (d_n^\alpha \Omega_{\alpha\beta}^\mu d_n^{+\beta} - c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_n^\beta) \sigma \\
 &+ l^2 p_\mu + \sum_{n=1}^{\infty} 2n (d_n^\alpha \Omega_{\alpha\beta}^\mu d_n^{+\beta} - c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_n^\beta) \tau \\
 &+ \frac{il}{2} \sum_{n=-\infty; n \neq 0}^{\infty} \frac{a_n^\mu}{n} e^{-2in(\tau+\sigma)}
 \end{aligned} \tag{6.21}$$

$$\begin{aligned}
 X^\mu &= x^\mu + l^2 p^\mu \tau + \frac{il}{2} \sum_{s=-\infty; s \neq 0}^{\infty} \frac{a_s^\mu}{s} e^{-2is(\tau+\sigma)} \\
 &+ \sum_{s=-\infty; s \neq 0; n=0; n+s \geq 0}^{\infty} \sum_{n=0}^{\infty} i \frac{2n+s}{2s} c_n^{+\alpha} \Omega_{\alpha\beta}^\mu c_{n+s}^\beta e^{-2is(\tau+\sigma)} \\
 &+ \sum_{s=-\infty; s \neq 0; n=0; n+s \leq -1}^{-1} \sum_{n=0}^{\infty} i \frac{2n+s}{2s} c_n^{+\alpha} \Omega_{\alpha\beta}^\mu d_{-n-s}^{+\beta} e^{-2is(\tau+\sigma)} \\
 &- \sum_{s=1}^{\infty} \sum_{n=1; s-n \geq 0}^{\infty} i \frac{2n-s}{2s} d_n^\alpha \Omega_{\alpha\beta}^\mu c_{s-n}^\beta e^{-2is(\tau+\sigma)} \\
 &- \sum_{s=-\infty; s \neq 0; n=1; n-s \geq 1}^{\infty} \sum_{n=1}^{\infty} i \frac{2n-s}{2s} d_n^\alpha \Omega_{\alpha\beta}^\mu d_{n-s}^{+\beta} e^{-2is(\tau+\sigma)}
 \end{aligned} \tag{6.22}$$

$$\Gamma \cdot p|\Psi\rangle = 0 \tag{6.23}$$

7 A Representation of the States of the Closed Supersymmetric String

The Case n Finite

As in [12], for n finite we have:

$$\begin{cases} a = -z; & a^+ = \frac{d}{dz} \\ c = \frac{d}{d\theta}; n & c^+ = \theta \\ d = \frac{d}{d\vartheta}; & d^+ = \vartheta \end{cases} \tag{7.1}$$

$$[a, a^+] = [c, c^+] = [d, d^+] = 1 \tag{7.2}$$

where $-z$ and d/dz are operators over CUET and θ and ϑ are Grassman variables with scalar product defined by:

$$\langle f, g \rangle = \int f(\theta)e^{\theta\theta^+} g^+(\theta) d\theta d\theta^+ \tag{7.3}$$

As for the bosonic string, a general state of the supersymmetric string can be written as:

$$\begin{aligned} \Psi_\alpha(x, \{z\}, \{\theta\}, \{\vartheta\}) = & [c_0 a_{\alpha 0}(x) + c(1, 0, 0) a_{\alpha \mu_1}^{i_1}(x) \vartheta_{i_1}^{\mu_1} \\ & + c(0, 1, 0) a_{\alpha \alpha_1}^{j_1}(x) \theta_{j_1}^{\alpha_1} + c(0, 0, 1) a_{\alpha \beta_1}^{k_1}(x) \vartheta_{k_1}^{\beta_1} + \dots \\ & + c(m, n, p) a_{\alpha \mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) \vartheta_{i_1}^{\mu_1} \dots \vartheta_{i_m}^{\mu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \\ & + \dots + \dots] \delta(\{z\}) \end{aligned} \tag{7.4}$$

where $c(m, n, p)$ are constants to be evaluated. In this case the physical state Ψ is a spinor whose components are defined in (7.4).

$$\Psi(x, \{z\}, \{\theta\}, \{\vartheta\}) = \begin{pmatrix} \Psi_1(x, \{z\}, \{\theta\}, \{\vartheta\}) \\ \Psi_2(x, \{z\}, \{\theta\}, \{\vartheta\}) \\ \vdots \\ \Psi_n(x, \{z\}, \{\theta\}, \{\vartheta\}) \end{pmatrix} \tag{7.5}$$

$$a_{\alpha \mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) = \begin{pmatrix} a_{1\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) \\ a_{2\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) \\ \vdots \\ a_{n\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) \end{pmatrix} \tag{7.6}$$

Its components are solutions of

$$\Gamma_\mu^{\beta\alpha} \partial^\mu \Phi_\alpha(x, \{z\}, \{\theta\}, \{\vartheta\}) = 0 \tag{7.7}$$

$$\Gamma_\mu^{\beta\alpha} \partial^\mu a_{\alpha \mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) = 0 \tag{7.8}$$

The Case $n \rightarrow \infty$

In this case:

$$\begin{cases} a = -z; & a^+ = -2z + \frac{d}{dz} \\ c = \frac{d}{d\theta}; & c^+ = \theta \\ d = \frac{d}{d\vartheta}; & d^+ = \vartheta \end{cases} \tag{7.9}$$

$$[a, a^+] = [c, c^+] = [d, d^+] = 1 \tag{7.10}$$

and the expression for the physical state of the string is similar to the finite case.

8 The Field of the Supersymmetric String

According to (6.17) and Sect. 7 the equation for the string field is given by

$$(\Gamma \cdot \partial)\Psi(x, \{z\}, \{\theta\}, \{\vartheta\}) = 0 \tag{8.1}$$

where $\{z\}$ denotes $(z_{1\mu}, z_{2\mu}, \dots, z_{n\mu}, \dots, \dots)$, and Ψ is a CUET in the set of variables $\{z\}$. Any UET of compact support can be written as a development of $\delta(\{z\})$ and its derivatives. Thus we have:

$$\begin{aligned} \Psi(x, \{z\}, \{\theta\}, \{\vartheta\}) = & [c_0 A_0(x) + c(1, 0, 0)A_{\mu_1}^{i_1}(x)\partial_{i_1}^{\mu_1} \\ & + c(0, 1, 0)A_{\alpha_1}^{j_1}(x)\theta_{j_1}^{\alpha_1} + c(0, 0, 1)A_{\beta_1}^{k_1}(x)\vartheta_{k_1}^{\beta_1} + \dots \\ & + c(m, n, p)A_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x)\partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \\ & + \dots + \dots] \delta(\{z\}) \end{aligned} \tag{8.2}$$

where the quantum fields $A_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x)$ are solutions of

$$(\Gamma \cdot \partial)A_{\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) = 0 \tag{8.3}$$

The propagator of the string field can be expressed in terms of the propagators of the component fields:

$$\begin{aligned} \Delta_{\alpha\beta}(x - x', \{z\}, \{z'\}, \{\theta\}, \{\theta'\}, \{\vartheta\}, \{\vartheta'\}) \\ = [c_0^2 \Delta_{\alpha\beta}(x - x') + \dots + c^2(m, n, p) \Delta_{\alpha\beta \mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p \nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p l_1 \dots l_m s_1 \dots s_n t_1 \dots t_p} \\ \times \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \partial_{l_1}^{\nu_1} \dots \partial_{l_m}^{\nu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \theta_{s_1}^{\gamma_1} \dots \theta_{s_n}^{\gamma_n} \\ \times \vartheta_{t_1}^{\delta_1} \dots \vartheta_{t_p}^{\delta_p} + \dots] \delta(\{z\}, \{z'\}) \end{aligned} \tag{8.4}$$

Writing

$$A_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(x) = \int_{-\infty}^{\infty} a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) e^{-ik_\mu x^\mu}$$

$$+ b_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{+i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) e^{ik_\mu x^\mu} d^{v-1} k \tag{8.5}$$

We may define the operators of annihilation and creation for a string as:

$$\begin{aligned} a_\alpha(k, \{z\}, \{\theta\}, \{\vartheta\}) &= [c_0 a_{0\alpha}(k) + c(1, 0, 0) a_{\alpha\mu_1}^{i_1}(k) \partial_{i_1}^{\mu_1} \\ &+ c(0, 1, 0) a_{\alpha j_1}^{j_1}(k) \theta_{j_1}^{\alpha_1} + c(0, 0, 1) a_{\alpha\beta_1}^{k_1}(k) \vartheta_{k_1}^{\beta_1} + \dots \\ &+ c(m, n, p) a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \\ &+ \dots + \dots] \delta(\{z\}) \end{aligned} \tag{8.6}$$

$$\begin{aligned} a_\alpha^+(k, \{z\}, \{\theta\}, \{\vartheta\}) &= [c_0 a_{0\alpha}^+(k) + c(1, 0, 0) a_{\alpha\mu_1}^{+i_1}(k) \partial_{i_1}^{\mu_1} \\ &+ c(0, 1, 0) a_{\alpha j_1}^{+j_1}(k) \theta_{j_1}^{+\alpha_1} + c(0, 0, 1) a_{\alpha\beta_1}^{+k_1}(k) \vartheta_{k_1}^{+\beta_1} + \dots \\ &+ c(m, n, p) a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{+i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \theta_{j_1}^{+\alpha_1} \dots \theta_{j_n}^{+\alpha_n} \vartheta_{k_1}^{+\beta_1} \dots \vartheta_{k_p}^{+\beta_p} \\ &+ \dots + \dots] \delta(\{z\}) \end{aligned} \tag{8.7}$$

where the constants $c(m, n, p)$ are solution of:

$$\begin{aligned} c^*(m, n, p) \vartheta_{k_p}^{+\beta_p} \dots \vartheta_{k_1}^{+\beta_1} \theta_{j_n}^{+\alpha_n} \dots \theta_{j_1}^{+\alpha_1} a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{+i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \\ = c(m, n, p) a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{+i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \theta_{j_1}^{+\alpha_1} \dots \theta_{j_n}^{+\alpha_n} \vartheta_{k_1}^{+\beta_1} \dots \vartheta_{k_p}^{+\beta_p} \end{aligned} \tag{8.8}$$

and define the creation and annihilation operators of the anti-string:

$$\begin{aligned} b_\alpha^+(k, \{z\}, \{\theta\}, \{\vartheta\}) &= [c_0 b_{0\alpha}^+(k) + c(1, 0, 0) b_{\alpha\mu_1}^{+i_1}(k) \partial_{i_1}^{\mu_1} \\ &+ c(0, 1, 0) b_{\alpha j_1}^{+j_1}(k) \theta_{j_1}^{\alpha_1} + c(0, 0, 1) b_{\alpha\beta_1}^{+k_1}(k) \vartheta_{k_1}^{\beta_1} + \dots \\ &+ c(m, n, p) b_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{+i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \\ &+ \dots + \dots] \delta(\{z\}) \end{aligned} \tag{8.9}$$

$$\begin{aligned} b_\alpha(k, \{z\}, \{\theta\}, \{\vartheta\}) &= [c_0 b_{0\alpha}(k) + c(1, 0, 0) b_{\alpha\mu_1}^{i_1}(k) \partial_{i_1}^{\mu_1} \\ &+ c(0, 1, 0) b_{\alpha j_1}^{j_1}(k) \theta_{j_1}^{\alpha_1} + c(0, 0, 1) b_{\alpha\beta_1}^{k_1}(k) \vartheta_{k_1}^{\beta_1} + \dots \\ &+ c(m, n, p) b_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k) \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \\ &+ \dots + \dots] \delta(\{z\}) \end{aligned} \tag{8.10}$$

As a consequence we have

$$\Psi_\alpha(x, \{z\}, \{\theta\}, \{\vartheta\}) = \int_{-\infty}^{\infty} a_\alpha(x, \{z\}, \{\theta\}, \{\vartheta\}) e^{-ik_\mu x^\mu}$$

$$+ b_\alpha^+(x, \{z\}, \{\theta\}, \{\vartheta\}) e^{ik_\mu x^\mu} d^{v-1} x \tag{8.11}$$

If we define

$$\begin{cases} [,]_{n+p+1} = [,] & n + p + 1 \text{ even} \\ [,]_{n+p+1} = \{ , \} & n + p + 1 \text{ odd} \end{cases} \tag{8.12}$$

with

$$\begin{aligned} & [a_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k), a_{\beta\nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}^{+l_1 \dots l_m s_1 \dots s_n t_1 \dots t_p}(k')]_{n+p+1} \\ & = f_{\alpha\beta\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p \nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p l_1 \dots l_m s_1 \dots s_n t_1 \dots t_p}(k) \delta(k - k') \end{aligned} \tag{8.13}$$

Then

$$\begin{aligned} & \{a_\alpha(k, \{z\}, \{\theta\}, \{\vartheta\}), a_\beta^+(k', \{z'\}, \{\theta'\}, \{\vartheta'\})\} \\ & = c_0^2 f_{0\alpha\beta}(k) + \dots + c^2(m, n, p) f_{\alpha\beta\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p \nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}(k - k') \\ & \quad \times \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \partial_{l_1}^{\nu_1} \dots \partial_{l_m}^{\nu_m} \theta_{j_1}^{\alpha_1} \dots \theta_{j_n}^{\alpha_n} \vartheta_{k_1}^{\beta_1} \dots \vartheta_{k_p}^{\beta_p} \theta_{s_1}^{+\gamma_1} \dots \theta_{s_n}^{+\gamma_n} \\ & \quad \times \vartheta_{t_1}^{\delta_1} \dots \vartheta_{t_p}^{\delta_p} + \dots] \delta(\{z\}, \{z'\}) \end{aligned} \tag{8.14}$$

and for the anti-string

$$\begin{aligned} & [b_{\alpha\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p}(k), b_{\beta\nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}^{+l_1 \dots l_m s_1 \dots s_n t_1 \dots t_p}(k')]_{n+p+1} \\ & = g_{\alpha\beta\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p \nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}^{i_1 \dots i_m j_1 \dots j_n k_1 \dots k_p l_1 \dots l_m s_1 \dots s_n t_1 \dots t_p}(k) \delta(k - k') \end{aligned} \tag{8.15}$$

Thus

$$\begin{aligned} & \{b_\alpha(k, \{z\}, \{\theta\}, \{\vartheta\}), b_\beta^+(k', \{z'\}, \{\theta'\}, \{\vartheta'\})\} \\ & = c_0^2 g_{0\alpha\beta}(k) + \dots + c^2(m, n, p) g_{\alpha\beta\mu_1 \dots \mu_m \alpha_1 \dots \alpha_n \beta_1 \dots \beta_p \nu_1 \dots \nu_m \gamma_1 \dots \gamma_n \delta_1 \dots \delta_p}(k) \delta(k - k') \\ & \quad \times \partial_{i_1}^{\mu_1} \dots \partial_{i_m}^{\mu_m} \partial_{l_1}^{\nu_1} \dots \partial_{l_m}^{\nu_m} \theta_{j_1}^{+\alpha_1} \dots \theta_{j_n}^{+\alpha_n} \vartheta_{k_1}^{+\beta_1} \dots \vartheta_{k_p}^{+\beta_p} \theta_{s_1}^{+\gamma_1} \dots \theta_{s_n}^{+\gamma_n} \\ & \quad \times \vartheta_{t_1}^{\delta_1} \dots \vartheta_{t_p}^{\delta_p} + \dots] \delta(\{z\}, \{z'\}) \end{aligned} \tag{8.16}$$

9 The Action for the Field of the Supersymmetric String

The Case n Finite

The action for the free supersymmetric closed string field is:

$$\begin{aligned} S_{free} & = \iiint_{\{\Gamma_1\}} \oint_{\{\Gamma_2\}} \int_{-\infty}^{\infty} i\bar{\Psi}(x, \{z_1\}, \{\theta_1\}, \{\vartheta_1\}) e^{\{z_1\} \cdot \{z_2\}} e^{\{\theta_1^+\} \cdot \{\theta_1\}} e^{\{\vartheta_1^+\} \cdot \{\vartheta_1\}} \\ & \quad \times \not{\partial} \Psi(x, \{z_2\}, \{\theta_1\}, \{\vartheta_1\}) d^v x \{dz_1\} \{dz_2\} \{d\theta_1^+\} \{d\theta_1\} \{d\vartheta_1^+\} \{d\vartheta_1\} \end{aligned} \tag{9.1}$$

where $\not{\partial} = \Gamma \cdot \partial$.

A possible interaction is given by:

$$\begin{aligned}
 S_{int} = & \lambda \iiint \iiint \iiint \oint_{\Gamma_1} \oint_{\Gamma_2} \oint_{\Gamma_3} \oint_{\Gamma_4} \int_{-\infty}^{\infty} \overline{\Psi}(x, \{z_1\}, \{\theta_1\}, \{\vartheta_1\}) \\
 & \times e^{\{z_1\} \cdot \{z_2\}} e^{\{\theta_1^+\} \cdot \{\theta_1\}} e^{\{\vartheta_1^+\} \cdot \{\vartheta_1\}} \\
 & \times \Psi(x, \{z_2\}, \{\theta_1\}, \{\vartheta_1\}) e^{\{z_2\} \cdot \{z_3\}} e^{\{\theta_1\} \cdot \{\theta_2^+\}} e^{\{\vartheta_1\} \cdot \{\vartheta_2^+\}} \\
 & \times \overline{\Psi}(x, \{z_3\}, \{\theta_2\}, \{\vartheta_2\}) e^{\{z_3\} \cdot \{z_4\}} e^{\{\theta_2^+\} \cdot \{\theta_2\}} e^{\{\vartheta_2^+\} \cdot \{\vartheta_2\}} \\
 & \times \Psi(x, \{z_4\}, \{\theta_2\}, \{\vartheta_2\}) d^v x \{dz_1\} \{dz_2\} \{dz_3\} \{dz_4\} \\
 & \times \{d\theta_1^+\} \{d\theta_1\} \{d\vartheta_1^+\} \{d\vartheta_1\} \{d\theta_2^+\} \{d\theta_2\} \{d\vartheta_2^+\} \{d\vartheta_2\}
 \end{aligned} \tag{9.2}$$

Both, S_{free} and S_{int} are non-local as expected.

The Case $n \rightarrow \infty$

In this case:

$$\begin{aligned}
 S_{free} = & \iiint \iiint \oint_{\Gamma_1} \oint_{\Gamma_2} \int_{-\infty}^{\infty} i \overline{\Psi}(x, \{z_1\}, \{\theta_1\}, \{\vartheta_1\}) e^{\{z_1\} \cdot \{z_2\}} e^{\{\theta_1^+\} \cdot \{\theta_1\}} e^{\{\vartheta_1^+\} \cdot \{\vartheta_1\}} \\
 & \times \not{\partial} \Psi(x, \{z_2\}, \{\theta_1\}, \{\vartheta_1\}) d^v x \{d\eta_1\} \{d\eta_2\} \{d\theta_1^+\} \{d\theta_1\} \{d\vartheta_1^+\} \{d\vartheta_1\}
 \end{aligned} \tag{9.3}$$

where

$$d\eta(z) = \frac{e^{-z^2}}{\sqrt{2} \pi} \tag{9.4}$$

and

$$\begin{aligned}
 S_{int} = & \lambda \iiint \iiint \iiint \oint_{\Gamma_1} \oint_{\Gamma_2} \oint_{\Gamma_3} \oint_{\Gamma_4} \int_{-\infty}^{\infty} \overline{\Psi}(x, \{z_1\}, \{\theta_1\}, \{\vartheta_1\}) \\
 & \times e^{\{z_1\} \cdot \{z_2\}} e^{\{\theta_1^+\} \cdot \{\theta_1\}} e^{\{\vartheta_1^+\} \cdot \{\vartheta_1\}} \\
 & \times \Psi(x, \{z_2\}, \{\theta_1\}, \{\vartheta_1\}) e^{\{z_2\} \cdot \{z_3\}} e^{\{\theta_1\} \cdot \{\theta_2^+\}} e^{\{\vartheta_1\} \cdot \{\vartheta_2^+\}} \\
 & \times \overline{\Psi}(x, \{z_3\}, \{\theta_2\}, \{\vartheta_2\}) e^{\{z_3\} \cdot \{z_4\}} e^{\{\theta_2^+\} \cdot \{\theta_2\}} e^{\{\vartheta_2^+\} \cdot \{\vartheta_2\}} \\
 & \times \Psi(x, \{z_4\}, \{\theta_2\}, \{\vartheta_2\}) d^v x \{d\eta_1\} \{d\eta_2\} \{d\eta_3\} \{d\eta_4\} \\
 & \times \{d\theta_1^+\} \{d\theta_1\} \{d\vartheta_1^+\} \{d\vartheta_1\} \{d\theta_2^+\} \{d\theta_2\} \{d\vartheta_2^+\} \{d\vartheta_2\}
 \end{aligned} \tag{9.5}$$

The convolution of two propagators of the string field is:

$$\hat{\Delta}_{\alpha\beta}(k, \{z_1\}, \{z_2\}, \{\theta_1\}, \{\theta'_1\}, \{\vartheta_1\}, \{\vartheta'_1\}) * \hat{\Delta}_{\alpha\beta}(k', \{z_3\}, \{z_4\}, \{\theta_2\}, \{\theta'_2\}, \{\vartheta_2\}, \{\vartheta'_2\}) \tag{9.6}$$

where $*$ denotes the convolution of Ultradistributions of Exponential Type on the k variable only. With the use of the result

$$\frac{1}{\rho} * \frac{1}{\rho} = -\pi^2 \ln \rho \tag{9.7}$$

($\rho = x_1^2 + x_2^2 + \dots + x_v^2$) in euclidean space and

$$\frac{1}{\rho \pm i0} * \frac{1}{\rho \pm i0} = \mp i \pi^2 \ln(\rho \pm i0) \quad (9.8)$$

($\rho = x_0^2 - x_1^2 - \dots - x_{v-1}^2$) in Minkowskian space, the convolution of two string field propagators is finite.

10 Discussion

We have decided to begin this paper, for the benefit of the reader, with a summary of the main characteristics of Ultradistributions of Exponential Type and their Fourier transform.

We have shown that UET are appropriate for the description in a consistent way superstring and superstring field theories. By means of a Lagrangian for the superstring we have obtained the non-linear Euler-Lagrange equations and solve them. We have given the movement equation for the field of the superstring and solve it with the use of CUET. We have shown that this superstring field is a linear superposition of CUET. We have evaluated the propagator for the superstring field, and calculate the convolution of two of them, taking into account that superstring field theory is a non-local theory of UET of an infinite number of complex variables. For practical calculations and experimental results we have given expressions that involve only a finite number of variables.

As a final remark we would like to point out that our formulae for convolutions follow from general definitions. They are not regularized expressions

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